TECHNICAL MEMORANDUM

Contract NAS8-20004

A NOTE ON THE ANALYTIC PLATFORM

by

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February 18, 1967

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I. INTRODUCTION

It was shown in the technical memorandum by James, Kelly, and Lowry [1] that coning motion about a reference axis in a rigid body could produce a net angular rotation about the reference axis even though $\int \omega_r \, dt = 0$, where ω_r is the angular velocity of the rigid body about the reference axis. To be more specific, if a rigid body is rotated about two axes perpendicular to each other and to the reference axis in such a manner that the reference axis generates a cone and then returns to its original alignment, then the rigid body will have rotated about the reference axis an amount equal to the area swept out on a unit sphere whose center is at the origin of the coordinate system of the body, while the $\int \omega_r \, dt = 0$. Since the ideal SAP measures the angle $\theta = \int \omega_r \, dt$, coning motion will cause the SAP to rotate about its reference axis even though this rotation will not show up in θ. Such is not the case with the analytic platform since the transformation is based on the rate of the change of the θ outputs of the SAP's, not on the θ's themselves.

In the report by Lowry [2], it was shown that the solution to a set of differential equations, knowing the exact angular velocity of a vehicle and initial orientation of the vehicle with respect to an inertial reference, would yield the new orientation of the vehicle. If the SAP's are exact, the rate of change of their θ

outputs will be the exact angular velocities about their reference axis. If the outputs of the SAP's are used to provide the exact angular velocities of the vehicle for the analytic platform, then the vehicle orientation will be known even though the SAP's could be in error due to coning motion.

One question concerning the analytic platform is the uniqueness of the solution, i. e., if the exact angular velocity of the vehicle is known, is the solution to the transformation differential equations unique? The answer is yes and uniqueness is proven in the following section.

II. UNIQUENESS OF TRANSFORMATION SOLUTION

Let

$$\dot{x}_i = f_i(x_1, x_2, \cdots, x_n, t) \ , \qquad i = 1, 2, \cdots, n \quad (1)$$

define a set of first order differential equations. Let $f_i(x_1, x_2, \cdots, x_n, t)$ be n real valued functions of the n+1 real variables $x_1, \cdots, x_n$, t defined and continuous* on an open region $\mathcal{L}$ of an (1+n)-dimensional euclidean space. If $f_i(x_1, x_2, \cdots, x_n, t)$ satisties Lipschitz conditions, presented in (2), then there exists a unique solution satisfying the given initial conditions. The formal theorem and its proof may be found in almost any standard text on advanced differential equations; the book by Nemytskii and Stepanov [3] presents an excellent, though formal, proof for the mathematician while the book by Murray and Miller [4] presents the proof in a more readable form for the engineer.

The Lipschitz condition is:

$$\begin{aligned} &|f_i(x_1^*, x_2^*, \cdots, x_n^*, t) - f_i(x_1^+, x_2^+, \cdots, x_n^+, t)| \\ &\qquad \leq L \sum_{i=1}^{n} |x_i^* - x_i^+|, \qquad i = 1, 2, \cdots, n \end{aligned} \quad (2)$$

for any two points $(x_1^*, x_2^*, \cdots, x_n^*, t)$ and $(x_1^+, x_2^+, \cdots, x_n^+, t)$ in $\mathcal{L}$. L is a constant and is referred to as the Lipschitz constant.

*That $f_i(x_i, x_2, \cdots, x_n, t)$ is continuous in x_i and t shall mean joint continuity; that is, given $\epsilon > 0$, there exists a $\delta > 0$, such that $|f_i(x_1, x_2, \cdots, x_n, t) - f_i(x_1', x_2', \cdots, x_n', t')| < \epsilon$ whenever $|x_i - x_i'| < \delta$ and $|t - t'| < \delta$.

One method of satisfying the Lipschitz conditions is to use the Law of the Mean [5,6]. By applying it to the functions in (1), the following results are obtained:

$$f_i(x_1^*, x_2^*, \cdots, x_n^*, t) - f_i(x_1^+, x_2^+, \cdots, x_n^+, t) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} (\bar{x}_i, \bar{x}_2, \cdots, \bar{x}_n, t) \ (x_j^* - x_j^+) \quad i = 1, 2, \cdots n, \tag{3}$$

where $\bar{x}_j$ lies between x_j^* and x_j^+. Taking the absolute value of (3),

$$|f_i(x_i^*, x_2^*, \cdots, x_n^*, t) - f_i(x_1^+, x_2^+, \cdots, x_n^+, t)| \leq \sum_{j=1}^{n} \left|\frac{\partial f_i}{\partial x_j}\right| |x_j^* - x_j^+| \ . \tag{4}$$

The $\frac{\partial f_i}{\partial x_j}$ are assumed to exist and to be continuous for $i, \ j = 1, \cdots, n$. If the $\frac{\partial f_i}{\partial x_j}$ are bounded, then there will be a L_{ij} such that

$$\left|\frac{\partial f_i}{\partial x_j}\right| \leq L_{ij}, \qquad i, j = 1, \cdots, n \ . \tag{5}$$

Let

$$L = \max_{i,j} \ L_{ij} \tag{6}$$

and then substitute (6) into (4).

$$|f_i(x_1^*, x_2^*, \cdots, x_n^*, t) - f_i(x_1^+, x_2^+, \cdots, x_n^+, t)| \leq L \sum_{j=1}^{n} |(x_j^* - x_j^+)|, \tag{7}$$

which is also the Lipschitz conditions.

From the report by Lowry [2], the following differential equations relating the transformation matrix (from a vehicle coordinate system to a space-fixed coordinate system) to the angular velocities of one of the coordiante systems (vehicle coordinate system) are presented below.

$$\begin{aligned} \dot{c}_{11} &= c_{12}\,\dot{\phi}_z - c_{13}\,\dot{\phi}_y \\ \dot{c}_{12} &= c_{13}\,\dot{\phi}_x - c_{11}\,\dot{\phi}_z \\ \dot{c}_{13} &= c_{11}\,\dot{\phi}_y - c_{12}\,\dot{\phi}_x \end{aligned} \tag{8a}$$

$$\begin{aligned} \dot{c}_{21} &= c_{22}\,\dot{\phi}_z - c_{23}\,\dot{\phi}_y \\ \dot{c}_{22} &= c_{23}\,\dot{\phi}_x - c_{21}\,\dot{\phi}_z \\ \dot{c}_{23} &= c_{21}\,\dot{\phi}_y - c_{22}\,\dot{\phi}_x \end{aligned} \tag{8b}$$

$$\begin{aligned} \dot{c}_{31} &= c_{32}\,\dot{\phi}_z - c_{33}\,\dot{\phi}_y \\ \dot{c}_{32} &= c_{33}\,\dot{\phi}_x - c_{31}\,\dot{\phi}_z \\ \dot{c}_{33} &= c_{31}\,\dot{\phi}_y - c_{32}\,\dot{\phi}_x \end{aligned} \tag{8c}$$

where

$$\dot{\phi} = \dot{\phi}_x\,\hat{i}_v + \dot{\phi}_y\,\hat{j}_v + \dot{\phi}_z\,\hat{k}_v \tag{9}$$

is the angular velocity of the vehicle and

$$\begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}. \tag{10}$$

By applying the Law of the Mean to the right-hand side of (8a), (8b), and (8c) and taking the absolute values, i. e., applying (4) to (8a), (8b), and (8c), the following equations may be obtained:

$$
\begin{aligned}
&|c_{12}^{*}\dot{\phi}_z - c_{13}^{*}\dot{\phi}_y - c_{12}^{+}\dot{\phi}_z + c_{13}^{*}\dot{\phi}_y| \leq |\dot{\phi}_z||c_{12}^{*} - c_{12}^{+}| + |\dot{\phi}_y||c_{13}^{*} - c_{13}^{+}| \\
&|c_{13}^{*}\dot{\phi}_x - c_{11}^{*}\dot{\phi}_z - c_{13}^{+}\dot{\phi}_x + c_{11}^{+}\dot{\phi}_z| \leq |\dot{\phi}_z||c_{11}^{*} - c_{11}^{+}| + |\dot{\phi}_x||c_{13}^{*} - c_{13}^{+}| \\
&|c_{11}^{*}\dot{\phi}_y - c_{12}^{*}\dot{\phi}_x - c_{11}^{+}\dot{\phi}_y + c_{12}^{+}\dot{\phi}_x| \leq |\dot{\phi}_y||c_{11}^{*} - c_{11}^{+}| + |\dot{\phi}_x||c_{12}^{*} - c_{12}^{+}| \\
&|c_{22}^{*}\dot{\phi}_z - c_{23}^{*}\dot{\phi}_y - c_{22}^{+}\dot{\phi}_z + c_{23}^{+}\dot{\phi}_y| \leq |\dot{\phi}_z||c_{22}^{*} - c_{22}^{+}| + |\dot{\phi}_y||c_{23}^{*} - c_{23}^{+}| \\
&|c_{23}^{*}\dot{\phi}_x - c_{21}^{*}\dot{\phi}_z - c_{23}^{+}\dot{\phi}_z + c_{21}^{+}\dot{\phi}_z| \leq |\dot{\phi}_z||c_{21}^{*} - c_{21}^{+}| + |\dot{\phi}_x||c_{22}^{*} - c_{22}^{+}| \\
&|c_{21}^{*}\dot{\phi}_y - c_{22}^{*}\dot{\phi}_x - c_{21}^{+}\dot{\phi}_y + c_{22}^{+}\dot{\phi}_y| \leq |\dot{\phi}_y||c_{21}^{*} - c_{21}^{+}| + |\dot{\phi}_x||c_{22}^{*} - c_{22}^{+}| \\
&|c_{32}^{*}\dot{\phi}_z - c_{33}^{*}\dot{\phi}_y - c_{32}^{+}\dot{\phi}_z + c_{33}^{+}\dot{\phi}_y| \leq |\dot{\phi}_z||c_{32}^{*} - c_{32}^{+}| + |\dot{\phi}_y||c_{33}^{*} - c_{33}^{+}| \\
&|c_{33}^{*}\dot{\phi}_x - c_{31}^{*}\dot{\phi}_z - c_{33}^{+}\dot{\phi}_x + c_{31}^{+}\dot{\phi}_z| \leq |\dot{\phi}_z||c_{31}^{*} - c_{31}^{+}| + |\dot{\phi}_x||c_{33}^{*} - c_{33}^{+}| \\
&|c_{31}^{*}\dot{\phi}_y - c_{32}^{*}\dot{\phi}_x - c_{31}^{+}\dot{\phi}_y + c_{32}^{+}\dot{\phi}_x| \leq |\dot{\phi}_y||c_{31}^{*} - c_{31}^{+}| + |\dot{\phi}_x||c_{32}^{*} - c_{32}^{+}|
\end{aligned}
\tag{11}
$$

The right-hand sides of (8a), (8b), and (8c) are continuous and bounded, and the angular velocities of the vehicle are continuous and bounded. Equation (11) is the Lipschitz condition if L is taken as the maximum value of $|\dot{\phi}_x|$, $|\dot{\phi}_y|$ or $|\dot{\phi}_z|$ and this value substituted into (11) in place of all the $|\dot{\phi}|$'s. Therefore, the soultions to (8a), (8b), and (8c) are unique for a given set of initial conditions.

REFERENCES

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